A Framework for the Hyperintensional Semantics of Natural Language with Two Implementations

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Abstract. In this paper we present a framework for constructing hyperintensional semantics for natural language. On this approach, the axiom of extensionality is discarded from the axiom base of a logic. Weaker conditions are specified for the connection between equivalence and identity which prevent the reduction of the former relation to the latter. In addition, by axiomatising an intensional number theory we can provide an internal account of proportional cardinality quantifiers, like *most*. We use a (pre-)lattice defined in terms of a (pre-)order that models the entailment relation. Possible worlds/situations/indices are then prime filters of propositions in the (pre-)lattice. Truth in a world/situation is then reducible to membership of a prime filter. We show how this approach can be implemented within (i) an intensional higher-order type theory, and (ii) first-order property theory.

1 Introduction

It has frequently been noted that the characterization of intensions as functions from possible worlds to extensions, as in [21], yields a semantics which is not sufficiently fine grained. Specifically, logically equivalent expressions are co-intensional and so intersubstitutable in all contexts, including the complements of propositional attitude predicates. This view of intensions defines them extensionally in set theoretic terms, and it has been dominant in formal semantics at least since Carnap [5].

An alternative view, which (following Lappin and Pollard [17]) we refer to as *hyperintensionalism*, posits propositions as independent intensional entities,

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and takes truth to be a relation between a proposition and a non-intensional entity. In the past twenty years a variety of hyperintensionalist theories have been proposed, including Thomason [28], situation semantics [3, 2, 25], Landman [16], property theory [6, 29, 32], [22], and Lappin and Pollard [17]. With the exception of Turner [32] these theories have focused on developing a model theory in which logical equivalence does not entail synonymy.

We depart from this tradition by taking an axiomatic approach to hyperintensionalism and assigning model theory a secondary role in our account. We define a class of models that provide a minimal model theory in which our logics are sound and which support counter-examples to the axiom of extensionality.

In section 2 we specify an intensional higher-order type theory in which the axioms of extensionality are replaced by weaker conditions on the relation between identity and equivalence. We introduce an equivalence (weak identity) predicate that can hold between any two expressions of the same type and use it to construct an intensional number theory. This theory permits us to add meaning postulates that characterize certain proportional cardinality quantifiers, like *most*, which have, until now, avoided an axiomatic treatment.

In section 3 we show how our axiomatic version of hyperintensionalism can be implemented in a first-order property theory where properties and other intensional entities are elements of a single multi-sorted domain. It is possible to mimic the higher-order type system of section 2 in this theory by defining appropriate sorting predicates for the domain. It is also possible to characterize S5 modalities in a straightforward way within this theory.

Section 4 gives an algebraic reduction of possible worlds (situations, indices) to prime filters in a (pre-)lattice of propositions.[1] This framework permits us to reduce truth in a hyperintensional semantics to membership in a prime filter of propositions in the (pre-)lattice. An appropriate (pre-)lattice can be constructed for the intensional higher-order type theory of section 2 and a full lattice for the first-order property theory of section 3.

Finally in section 5 we compare our account to other views of hyperintensionalism that have been proposed in the recent literature.

2 An Intensional Higher-Order Type Theory

2.1 The Proof Theory

We define the set of types in our intensional higher-order type theory IHTT as follows.

Basic Types

1. e (individuals)
2. Π (propositions)

[1] This idea is due to Carl Pollard, and it is presented in [17]

Exponential Types

If A, B are types, then A^B is a type.

This is the type system of Church [7], which is equivalent to Ty2 of Gallin [13] without s (possible worlds) as a basic type. For each type A there is (i) a denumerable set of non-logical constants of type A and (ii) a denumerable set of variables of type A. We define the set E_A of expressions of type A as follows.

1. Every variable of type A is in E_A.
2. Every constant of type A is in E_A.
3. If $\alpha \in E_A$ and u is a variable in E_B, then $\lambda u\alpha \in E_{A^B}$.
4. If $\alpha \in E_{B^A}$ and $\beta \in E_A$, then $\alpha(\beta) \in E_B$.
5. If $\alpha, \beta \in E_A$, then $\alpha = \beta \in E_\Pi$.
6. if $\alpha, \beta \in E_A$, then $\alpha \cong \beta \in E_\Pi$.
7. $\top$ and $\bot \in E_\Pi$.
8. If $\phi, \psi \in E_\Pi$, then so are
   (a) $\neg\phi$
   (b) $\phi \vee \psi$
   (c) $\phi \wedge \psi$
   (d) $\phi \rightarrow \psi$
   (e) $\phi \leftrightarrow \psi$
9. If $\phi \in E_\Pi$ and u is a variable in E_A, then $\forall u\phi$ and $\exists u\phi \in E_\Pi$.

We adopt the following axioms, based, in part, on the higher-order intuitionistic type theory presented by Lambek and Scott [15, Part II, Chapter 1].[2]

(IHTT1) $p \vdash \top$
(IHTT2) $\bot \vdash p$
(IHTT3) $\vdash \neg p \leftrightarrow p \rightarrow \bot$
(IHTT4) $r \vdash p \wedge q$ iff $r \vdash p$ and $r \vdash q$
(IHTT5) $p \vee q \vdash r$ iff $p \vdash r$ or $q \vdash r$
(IHTT6) $p \vdash q \rightarrow r$ iff $p \wedge q \vdash r$
(IHTT7) $p \vdash \forall x_{\in B}\phi_{\in \Pi^B}$ iff $p \vdash \phi$
(IHTT8) $\phi(a) \vdash \exists x_{\in B}\phi(x)$ (where $\phi \in \Pi^B$, and a is a constant in B)
(IHTT9) $\vdash \lambda u\phi(v) \cong \phi^{u/v}$ (where u is a variable in A, $v \in$ A, $\phi \in B^A$, and v is not bound when substituted for u in ϕ)
(IHTT10) $\vdash \forall s, t_{\in \Pi}(s \cong t \leftrightarrow (s \leftrightarrow t))$
(IHTT11) $\vdash \forall \phi, \psi_{\in B^A}(\forall u_{\in A}(\phi(u) \cong \psi(u)) \rightarrow \phi \cong \psi)$ [3]
(IHTT12) $\vdash \forall u, v_{\in A}\forall \phi_{\in B^A}(u = v \rightarrow \phi(u) \cong \phi(v))$

[2] we are grateful to Carl Pollard for proposing the strategy of constructing an intensional higher order type system by starting with the Lambek-Scott system and discarding the axiom of extensionality, and for helpful suggestions on how to specify the relation between = and $\cong$.

[3] As Carl Pollard points out to us, axiom (11) together with axiom (10) entail that $\cong$ is extensional, and hence is an equivalence relation for propositions and predicates. We assume the axioms which specify that = (logical identity) is an equivalence relation.

(IHTT13) $\vdash \forall t_{\in \Pi}(t \vee \neg t)$

Axioms (IHTT1)–(IHTT12) yield a higher-order intuitionistic logic. We can obtain a classical Boolean system by adding axiom (IHTT13).

The relation $\cong$ corresponds to extensional equivalence of entities of the same type.[4] It follows from axiom (IHTT12) that logical identity implies equivalence. However, as the converse (the axiom of extensionality) does not hold, any two entities of the same type can be equivalent but not identical. Therefore, two expressions can be logically equivalent but not co-intensional. Specifically, it is possible for two propositions to be provably equivalent but distinct. Axiom (IHTT9) insures that two sides of a lambda conversion are logically equivalent, but it does not entail identical.

2.2 A Class of Possible Models

A model for IHTT is an ordered quintuple $\langle D, S, L, I, F \rangle$, where D is a family of non-empty sets such that each D_A is the set of possible denotations for expressions of type A, S and L are non-empty sets and $L \subset S$. I is a function from the expressions of IHTT to S such that if α is a non-logical constant, then $I(\alpha) \in L$; otherwise, $I(\alpha) \in S - L$. F is a function from L to members of D. If α is a non-logical constant in A, then $F(I(\alpha)) \in D_A$. $D_\Pi = \{t, f\}$. I assigns intensions to the expressions of IHTT, and F assigns denotations to the non-logical constants. A valuation g is a function from the variables of IHTT to members of D such that for each variable $v_{\in A} g(v) \in D_A$.

1. If $\alpha_{\in A}$ is a non-logical constant, then $\|\alpha\|^{M,g} = F(I(\alpha))$.
2. If $\alpha_{\in A}$ is a variable, then $\|\alpha\|^{M,g} = g(\alpha)$.
3. $\|\alpha_{\in B^A}(\beta_{\in A})\|^{M,g} = \|\alpha\|^{M,g}(\|\beta\|^{M,g})$.
4. If α is in A and u is a variable in B, then $\|\lambda u \alpha\|^{M,g}$ is a function $h \in (D_B)_A^D$ such that for any $a \in D_A, h(a) = \|\alpha\|^{M,g(u/a)}$.
5. $\|\neg \phi_{\in \Pi}\|^{M,g} = t$ iff $\|\phi\|^{M,g} = f$.
6. $\|\phi_{\in \Pi} \wedge \psi_{\in \Pi}\|^{M,g} = t$ iff $\|\phi\|^{M,g} = \|\psi\|^{M,g} = t$.
7. $\|\phi_{\in \Pi} \vee \psi_{\in \Pi}\|^{M,g} = t$ iff $\|\phi\|^{M,g} = t$ or $\|\psi\|^{M,g} = t$.
8. $\|\phi_{\in \Pi} \rightarrow \psi_{\in \Pi}\|^{M,g} = t$ iff $\|\phi\|^{M,g} = f$ or $\|\psi\|^{M,g} = t$.
9. $\|\phi_{\in \Pi} \leftrightarrow \psi_{\in \Pi}\|^{M,g} = t$ iff $\|\phi\|^{M,g} = \|\psi\|^{M,g}$.
10. $\|\alpha_{\in A} \cong \beta_{\in A}\|^{M,g} = t$ iff $\|\alpha\|^{M,g} = \|\beta\|^{M,g}$.
11. $\|\alpha_{\in A} = \beta_{\in A}\|^{M,g} = t$ iff $I(\alpha) = I(\beta)$.
12. $\|\forall u_{\in A} \phi_{\in \Pi}\|^{M,g} = t$ iff for all $a \in D_A$ $\|\phi\|^{M,g(u/a)} = t$.
13. $\|\exists u_{\in A} \phi_{\in \Pi}\|^{M,g} = t$ iff for some $a \in D_A$ $\|\phi\|^{M,g(u/a)} = t$.
14. $\phi_{\in \Pi}$ is true in M (false in M) iff $\|\phi\|^{M,G} = t(f)$ for all g.
15. $\phi_{\in \Pi}$ is logically true (false) iff ϕ is true (false) in every M.
16. $\phi_{\in \Pi} \models \psi_{\in \Pi}$ iff for every M such that ϕ is true in M, ψ is true in M.

[4] Within the framework of program specification theory, Maibaum [19] discusses the use of a weak non-logical equality predicate to express the equivalence/congruence of possibly distinct expressions within a theory.

The axioms (IHTT1)–(IHTT14) hold in models which satisfy these conditions. Notice that while $\alpha \cong \beta$ is true (relative to M, g) iff the denotations of α and β are the same, $\alpha = \beta$ is true iff I assigns α and β the same value. For a model M $I(\alpha) = I(\beta)$ implies that $\|\alpha\|^{M,g} = \|\beta\|^{M,g}$, for all g, but the converse implication does not hold. Therefore, it is possible for $\|\alpha \cong \beta\|^{M,g}$ to be true for all M, g, but for there to be at least one M and one g such that $\|\alpha = \beta\|^{M,g}$ is false. This would be the case, for example, if α and β were logically equivalent but distinct sentences in IHTT, and in at least one M, I is a function from the expressions of IHTT to their Gödel numbers.

2.3 Hyperintensional Number Theory and Proportional Cardinality Quantifiers

We add a new basic type N to our type system.[5] By substituting $\cong$ for $=$ in the Peano axioms, we can construct a hyperintensional number theory within IHTT.

(IHTT14) $\vdash \neg\exists u_{\in N}(succ(u) \cong 0)$
(IHTT15) $\vdash \forall u, v_{\in N}(succ(u) \cong succ(v) \leftrightarrow u \cong v)$
(IHTT16) $\vdash \forall \alpha_{\in \Pi^N}(\alpha(0) \wedge \forall u_{\in N}((\alpha(u) \rightarrow \alpha(succ(u))) \rightarrow \forall v_{\in N}\alpha(v))$

The basic arithmetical operations are defined in the usual way, but with $\cong$ substituted for $=$. In this theory it is possible for distinct representations of a number to be equivalent but not identical. Therefore. $7 + 2 \cong 9$, but it not necessarily the case that $7 + 2 = 9$.

We can specify the relation $<$ by means of the following axiom.

(IHTT17) $\vdash \forall u, v_{\in N}(u < v \leftrightarrow \exists w_{\in N}(\neg(w \cong 0) \wedge u + w \cong v))$

Let P be a property term in Π^A. We characterize the cardinality of P, $|P|$, by the following axioms.

(IHTT18) $\vdash \neg\exists u_{\in A}P(u) \rightarrow |P| \cong 0$
(IHTT19) $\vdash P(u) \rightarrow (\neg P^{-u}(u) \wedge \forall v \in A((\neg(u \cong v) \wedge P(v)) \rightarrow P^{-u}(v)))$
(IHTT20) $\vdash P(u) \rightarrow |P| \cong |P^{-u}| + succ(0)$

Take *most* to be a generalized quantifier of type $\Pi^{(\Pi^A)^{(\Pi^A)}}$. We give the interpretation of *most* by the following axiom.

(IHTT21) $\vdash \forall P, Q_{\in \Pi^A}(most(P)(Q)) \leftrightarrow |\lambda u[P(u) \wedge Q(u)]| > |\lambda u[P(u) \wedge \neg Q(u)|$

[5] We are grateful to Tom Maibaum for suggesting this approach to the internal representation of generalized quantifiers involving cardinality relations

3 A Property-theoretic Approach

The version of Property Theory presented here is Turner's weak first-order axiomatisation of Aczel's Frege Structures [30, 31, 32, 1]. Rather than avoiding the logical paradoxes of self-application through strong typing, here self-application is permitted, but the theory is deliberately too weak to allow the axioms of truth to apply to pathological cases of self application. Natural language types can be reintroduced as first-order sorts [30, 31, 32].

The methodology of adopting weak axioms to avoid category mistakes can be extended to cases of infelicitous references that arise with both non-denoting definites and anaphora [9, 11, 10, 12, 8].

Here we sketch two extensions to the theory. In the first, we introduce possible worlds to allow for a treatment of doxastic modality. This helps to highlight the distinction between intensionality and modality. In the second extension, we indicate how the theory can be strengthened to give a theory internal analysis of proportional cardinality quantifiers, similar to the one given in section 2.3 but without leaving an essentially first-order regime.

3.1 Basic Property Theory

A highly intensional theory of propositions and truth will allow distinct propositions to be (necessarily) true together. One such theory is Turner's axiomatisation of a Frege Structure [31, 1]; one of a family of theories known as Property Theory. It is a first-order theory with weak typing and consists of a language of terms, in which intensional propositions, properties and relations are represented, and a language of well formed formulæ, in which the (extensional) truth conditions of propositions can be expressed. To avoid a logical paradox, only some terms correspond to propositions, and hence have truth conditions.

In this version of first-order Property Theory, the language of terms is that of the untyped λ-calculus. This gives us a ready-made notion of predication—that of function application[6]—as well as providing the expressivity required for a compositional theory of natural language semantics.

(PT1) $t ::= c \mid x \mid \lambda x(t) \mid t(t) \mid l$ (The language of the untyped λ-calculus—individual constants, variables, abstractions, and applications of terms—together with logical constants.)

(PT2) $l ::= \hat{\wedge} \mid \hat{\vee} \mid \hat{\neg} \mid \hat{\rightarrow} \mid \hat{\forall} \mid \hat{\exists} \mid \hat{=}$ (The logical constants, corresponding with conjunction, disjunction, negation, universal quantification, existential quantification and equality.)

The language of terms is governed by the usual axioms for the untyped λ-calculus, namely those for α and β reduction. These define the equality of terms.

(λ1) $\lambda x(t) = \lambda y(t[y/x])$ if y not free in t. (α-reduction.)

[6] Note that there are cases where it is arguably inappropriate to equate predication with function application [4].

(λ2) $(\lambda x(t))t' = t[t'/x]$ (β-reduction.)

The truth conditions of those terms corresponding to propositions can be obtained in the first-order language of well formed formulæ (wff).

(PT3) $\varphi ::= t = t \mid \mathsf{Prop}(t) \mid \mathsf{True}(t) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \neg\varphi \mid \forall x(\varphi) \mid \exists x(\varphi)$ (The language of wff consists of statements of equality between terms, assertions that a term is a proposition, and that a term is a true proposition, together with the usual logical connectives.)

The predicate Prop is used to restrict the axioms of True to terms of the appropriate category. In particular, we cannot consider the truth conditions of terms that correspond with paradoxical statements.

The predicate Prop is governed by the following axioms.

(P1) $(\mathsf{Prop}(s) \wedge \mathsf{Prop}(t)) \rightarrow \mathsf{Prop}(s \hat{\wedge} t)$
(P2) $(\mathsf{Prop}(s) \wedge \mathsf{Prop}(t)) \rightarrow \mathsf{Prop}(s \hat{\vee} t)$
(P3) $\mathsf{Prop}(t) \rightarrow \mathsf{Prop}(\hat{\neg} t)$
(P4) $(\mathsf{Prop}(s) \wedge (\mathsf{True}(s) \rightarrow \mathsf{Prop}(t))) \rightarrow \mathsf{Prop}(s \hat{\rightarrow} t)$
(P5) $\forall x(\mathsf{Prop}(t)) \rightarrow \mathsf{Prop}(\hat{\forall} x(t))$
(P6) $\forall x(\mathsf{Prop}(t)) \rightarrow \mathsf{Prop}(\hat{\exists} x(t))$
(P7) $\mathsf{Prop}(s \hat{=} t)$

It only remains to have axioms for a theory of truth. In a theory without possible worlds, the following is sufficient.

(T1) $(\mathsf{Prop}(s) \wedge \mathsf{Prop}(t)) \rightarrow (\mathsf{True}(s \hat{\wedge} t) \leftrightarrow (\mathsf{True}(s) \wedge \mathsf{True}(t)))$
(T2) $(\mathsf{Prop}(s) \wedge \mathsf{Prop}(t)) \rightarrow (\mathsf{True}(s \hat{\vee} t) \leftrightarrow (\mathsf{True}(s) \vee \mathsf{True}(t)))$
(T3) $\mathsf{Prop}(t) \rightarrow (\mathsf{True}(\hat{\neg} t) \leftrightarrow \neg \mathsf{True}(t))$
(T4) $(\mathsf{Prop}(s) \wedge (\mathsf{True}(s) \rightarrow \mathsf{Prop}(t))) \rightarrow (\mathsf{True}(s \hat{\rightarrow} t) \leftrightarrow (\mathsf{True}(s) \rightarrow \mathsf{True}(t)))$
(T5) $\forall x(\mathsf{Prop}(t)) \rightarrow (\mathsf{True}(\hat{\forall} x(t)) \leftrightarrow \forall x(\mathsf{True}(t)))$
(T6) $\forall x(\mathsf{Prop}(t)) \rightarrow (\mathsf{True}(\hat{\exists} x(t)) \leftrightarrow \exists x(\mathsf{True}(t)))$
(T7) $\mathsf{True}(s \hat{=} t) \leftrightarrow (s = t)$
(T8) $\mathsf{True}(p) \rightarrow \mathsf{Prop}(p)$

We can define a notion of property, Pty.

(PT4) $\mathsf{Pty}(t) =_{\mathsf{def}} \forall x(\mathsf{Prop}(t(x)))$ (Properties are those terms that form a proposition with any term.)

Functional types can be defined in the language of wff.

(PT5) $(Q \Longrightarrow R)(t) =_{\mathsf{def}} \forall x(Q(x) \rightarrow R(tx))$

A type of the form $\langle a, b \rangle$ in Montague's **IL** would be written as $(a \Longrightarrow b)$. However, unlike in **IL**, a term can belong to more than one type.

If we are interested in characterising possible worlds in this intensional theory of propositions, one way of proceeding is to formulate a notion of truth with respect to an information state corresponding to a possible world. We shall try and see how much of the theory can be internally defined, where internal definability (of predicates) is characterised as follows:

(PT6) A predicate P is *internally definable* iff there exists a property p, in the language of terms, such that $\forall x(P(x) \leftrightarrow \mathsf{True}(px))$ [31, 1].

Not all predicates are internally definable. In particular, we cannot introduce properties corresponding to $\mathsf{Prop}, \mathsf{True}$ without obtaining a paradox [31, 1]. We shall see how much this restricts our ability to internally define a theory of possible worlds in a Frege Structure. In effect, if all the relevant notions were internally definable, then possible worlds would be a conservative extension of a Frege Structure. In other words, a Frege Structure would then already give us what we want, and the theory of possible worlds would be mere syntactic sugar.

In order to provide a familiar notation, a Fregean notion of set and set membership can be defined:

(PT7) $\{x \mid t\} =_{\mathsf{def}} \lambda x(t)$ (A *Fregean set* is sugar for a λ-abstract.)
(PT8) $(t \in s) =_{\mathsf{def}} \mathsf{True}(s(t))$ (Set membership is sugar for asserting that a property holds of a term.)

As we can see, the language of wff is strictly first-order. Using the untyped λ-calculus for the language of terms means that there might be some complexity in determining the equality of terms, but this is really no different from any other first-order theory: the standard way of classifying the formal power of a logic makes no claim about the complexity of determining the truth of atomic propositions.

There are many other approaches to intensionality that incorporate, or at least mimic two notions of equality. One that has been used in the context of linguistics is a semantics based on Martin-Löf's Type Theory (MLTT) [20, 27, 23, 24]. This is essentially constructive in nature, unlike the classical Property Theory presented here. One weakness of pure MLTT in relation to the semantics of natural language is that intensionality collapses in the case of false propositions. Its strength lies in the analysis of the dynamics of language. Because MLTT can be embedded in a Frege Structure [26], this approach to dynamics can be exploited and adapted in various ways within Property Theory [10, 8].

3.2 Property-theoretic Number Theory, and Proportional Cardinality Quantifiers

As the language of terms of PT is the λ-calculus, the most straightforward way of incorporating numbers into the theory might appear to be to adopt the appropriate the standard untyped λ-calculus definitions for $succ, pred, zero$. There are problems in doing so: the notion of identity in the theory is that of the λ-calculus; terms that are λ-equivalent are indistinguishable, yet we do not necessarily wish to force terms to be identical if they happen to have the same arithmetic evaluation.[7]

[7] These are similar to the arguments against using the λ-internal definitions for truth, false and the logical connectives for the interpretation of the intensional connectives of PT.

Instead, to the language of terms we add $succ, pred, add, mult, 0$ (for the successor, predecessor, addition and multiplication of terms representing natural numbers, and zero, respectively). And to the language of wff, we add Zero, for testing if a term is 0, $=_\eta$ for equality of numeric terms, and Num, a new category corresponding with the natural numbers. So now our language of terms is as follows:

$$t ::= x \mid c \mid tt \mid \lambda x.t \mid \hat{\wedge} \mid \hat{\vee} \mid \hat{\rightarrow} \mid \hat{\neg} \mid \hat{\forall} \mid \hat{\exists} \mid \hat{=} \mid succ \mid pred \mid add \mid mult \mid 0$$

And the language of wff is:

$$\varphi ::= t = t \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \neg\varphi \mid \forall x\varphi \mid \exists x\varphi \mid \mathsf{Zero} \mid=_\eta\mid \mathsf{Num}$$

Num has the following closure axioms (axiom schema):

($\mathrm{N}_{\mathsf{PT}}1$) $\mathsf{Num}(0) \wedge \forall y(\mathsf{Num}(y) \rightarrow \mathsf{Num}(succ(y)))$
($\mathrm{N}_{\mathsf{PT}}2$) $(\phi[0] \wedge \forall y(\mathsf{Num}(y) \wedge \phi[y] \rightarrow \phi[succ(y)])) \rightarrow \forall x(\mathsf{Num}(x) \rightarrow \phi[x])$

These give the basic closure axiom on 0 and its successors, together with the weak induction axiom scheme. Now we introduce axioms corresponding to the usual basic formulation of Peano arithmetic:

($\mathrm{N}_{\mathsf{PT}}3$) $\neg\exists x(\mathsf{Num}(x) \wedge succ(x) =_\eta 0)$
($\mathrm{N}_{\mathsf{PT}}4$) $\mathsf{Num}(x) \wedge \mathsf{Num}(y) \rightarrow (succ(x) =_\eta succ(y) \leftrightarrow x =_\eta y)$
($\mathrm{N}_{\mathsf{PT}}5$) $\mathsf{Zero}(0)$
($\mathrm{N}_{\mathsf{PT}}6$) $\mathsf{Num}(x) \rightarrow \neg\mathsf{Zero}(succ(x))$
($\mathrm{N}_{\mathsf{PT}}7$) $\mathsf{Num}(x) \rightarrow pred(succ(x)) =_\eta x$

Here, we not introduce an explicit term for $\perp$, and instead have axioms that are deliberately too weak to derive anything in cases where $\perp$ would otherwise arise.

The operations of addition, multiplication and equality can also be axiomatised:

($\mathrm{N}_{\mathsf{PT}}8$) $\mathsf{Num}(x) \rightarrow add(x)(0) =_\eta x$
($\mathrm{N}_{\mathsf{PT}}9$) $\mathsf{Num}(x) \wedge \mathsf{Num}(y) \rightarrow add(x)(succ(y)) =_\eta succ(add(x)(y))$
($\mathrm{N}_{\mathsf{PT}}10$) $\mathsf{Num}(x) \rightarrow mult(x)(0) =_\eta x$
($\mathrm{N}_{\mathsf{PT}}11$) $\mathsf{Num}(x) \wedge \mathsf{Num}(y) \rightarrow mult(x)(succ(y)) =_\eta add(mult(x)(y))(x)$
($\mathrm{N}_{\mathsf{PT}}12$) $\mathsf{Num}(y) \rightarrow (0 =_\eta y \leftrightarrow \mathsf{Zero}(y))$
($\mathrm{N}_{\mathsf{PT}}13$) $\mathsf{Num}(x) \rightarrow (x =_\eta 0 \leftrightarrow \mathsf{Zero}(x))$

We can define the notion of $<$:

($\mathrm{N}_{\mathsf{PT}}14$) $x < y =_{\mathsf{def}} \mathsf{Num}(x) \wedge \mathsf{Num}(y) \wedge \exists z(\mathsf{Num}(z) \wedge \neg\mathsf{Zero}(z) \wedge add(x)(z) =_\eta y)$

Equivalently, we can axiomatise it as follows:

1. $\mathsf{Num}(y) \rightarrow 0 < y$
2. $\mathsf{Num}(x) \rightarrow x \not< 0$
3. $\mathsf{Num}(x) \wedge \mathsf{Num}(y) \rightarrow (succ(x) < succ(y) \leftrightarrow x < y)$

We can internalise the natural number type and its relations. To the language of terms we add $\eta, \zeta, \hat{=}_\eta$ corresponding with $\mathsf{Num}, \mathsf{Zero}, =_\eta$.

(N$_{\mathsf{PT}}$15) $\mathsf{Prop}(\eta t)$
(N$_{\mathsf{PT}}$16) $\mathsf{True}(\eta t) \leftrightarrow \mathsf{Num}(t)$
(N$_{\mathsf{PT}}$17) $\mathsf{Num}(t) \rightarrow \mathsf{Prop}(\zeta(t))$
(N$_{\mathsf{PT}}$18) $\mathsf{Num}(t) \wedge \mathsf{Num}(s) \rightarrow \mathsf{Prop}(t \hat{=}_\eta s)$
(N$_{\mathsf{PT}}$19) $\mathsf{Num}(t) \rightarrow (\mathsf{True}(\zeta(t)) \leftrightarrow \mathsf{Zero}(t))$
(N$_{\mathsf{PT}}$20) $\mathsf{Num}(t) \wedge \mathsf{Num}(s) \rightarrow (\mathsf{True}(t \hat{=}_\eta s) \leftrightarrow t =_\eta s)$

The term $\hat{<}$ can be defined by:

(N$_{\mathsf{PT}}$21) $x \hat{<} y =_{\mathsf{def}} \eta x \hat{\wedge} \eta y \hat{\wedge} \hat{\exists} z (\eta z \hat{\wedge} \hat{\neg} \zeta(z) \hat{\wedge} add(x)(z) \hat{=}_\eta y)$

"Most" in $\mathbf{PT}_n$ The treatment of cardinality and *most* in PT is parallel to the accounts given in IHTT. Cardinality of properties can be given by:

(N$_{\mathsf{PT}}$23) $\mathsf{Pty}(p) \wedge \neg \exists x \mathsf{True}(px) \rightarrow |p| =_\eta 0$
(N$_{\mathsf{PT}}$24) $\mathsf{Pty}(p) \wedge \mathsf{True}(px) \rightarrow |p| =_\eta add(|p - \{x\}|)(succ(0))$

where $|\cdot|$ is a new term, and $-, \{\cdot\}$ are defined in the usual Property-theoretic manner.

Now we can axiomatise *most* by way of the following:

$$\mathsf{Pty}(p) \wedge \mathsf{Pty}(q) \rightarrow \mathsf{Prop}(most(p)(q))$$
$$\mathsf{Pty}(p) \wedge \mathsf{Pty}(q) \rightarrow \mathsf{True}(most(p)(q)) \leftrightarrow |\{x : px \hat{\wedge} \hat{\neg} qx\}| < |\{x : px \hat{\wedge} qx\}|$$

However, give our internalisation of the natural numbers, "most" can be made definitional:

$$most(p)(q) =_{\mathsf{def}} |\{x : px \hat{\wedge} \hat{\neg} qx\}| \hat{<} |\{x : px \hat{\wedge} qx\}|$$

This theory is substantially theorem equivalent with the IHTT version of intensional arithmetic.

3.3 Possible Worlds with Intentional Propositions

We can take information states to be sets of propositions.[8] We will use the terms $\mathfrak{w}, \mathfrak{u}, \mathfrak{v}, \ldots$ to denote information states. We can define the family of information states to be W. We will add W to the language of wff. The terms $\mathfrak{w}, \mathfrak{u}, \mathfrak{v}, \ldots$ are constants which correspond with prime filters over propositions.

We can now axiomatise a theory of truth with respect to information states, or prime filters over propositions. We will write $\mathsf{W}(\mathfrak{w})$ when $\mathfrak{w}$ is an information state corresponding to a possible world, and $p \in \mathfrak{w}$ when the proposition p holds in the information state $\mathfrak{w}$.

[8] In section 4 we formulate the reduction of possible worlds to set of propositions algebraically, where the sets of propositions are prime filters on the lattice of propositions generated by the logic of PT. For IHTT we require a pre-lattice with a pre-order rather than a lattice.

(PW1) $(\mathsf{W}(\mathfrak{w}) \wedge \mathsf{Prop}(p)) \rightarrow \mathsf{Prop}(\mathfrak{w}(p)))$
(PW2) $(\mathsf{W}(\mathfrak{w}) \wedge \mathsf{Prop}(p) \wedge \mathsf{Prop}(q)) \rightarrow ((p \hat{\wedge} q) \in \mathfrak{w} \leftrightarrow (p \in \mathfrak{w} \wedge q \in \mathfrak{w}))$
(PW3) $\mathsf{W}(\mathfrak{w}) \rightarrow ((p \hat{\vee} q) \in \mathfrak{w} \leftrightarrow (p \in \mathfrak{w} \vee q \in \mathfrak{w}))$
(PW4) $\mathsf{W}(\mathfrak{w}) \rightarrow ((p \hat{\rightarrow} q) \in \mathfrak{w} \leftrightarrow (p \in \mathfrak{w} \rightarrow q \in \mathfrak{w}))$
(PW5) $(\mathsf{W}(\mathfrak{w}) \wedge \mathsf{Prop}(p)) \rightarrow ((\hat{\neg} p) \in \mathfrak{w} \leftrightarrow \neg(p \in \mathfrak{w}))$
(PW6) $(\mathsf{W}(\mathfrak{w}) \wedge \mathsf{Prop}(p) \wedge (p \in \mathfrak{w} \rightarrow \mathsf{Prop}(q))) \rightarrow ((p \hat{\rightarrow} q) \in \mathfrak{w} \leftrightarrow (p \in \mathfrak{w} \rightarrow q \in \mathfrak{w}))$
(PW7) $(\mathsf{W}(\mathfrak{w}) \wedge \forall x(\mathsf{Prop}(p))) \rightarrow ((\hat{\forall} x(p)) \in \mathfrak{w} \leftrightarrow \forall x(p \in \mathfrak{w}))$
(PW8) $\mathsf{W}(\mathfrak{w}) \rightarrow ((\hat{\exists} x(p)) \in \mathfrak{w} \leftrightarrow \exists x(p \in \mathfrak{w}))$

The axiom (PW1) is concerned with the felicity of propositions when $\mathfrak{w}$ is used as a proposition. Essentially it encapsulates a typing constraint that we are only concerned with whether or not a proposition is in the prime filter $\mathfrak{w}$. The remaining axioms give $\mathfrak{w}$ the properties of a prime filter.

In effect, each $\mathfrak{w}$ in W is a different theory. We have generalised the axioms of truth in Property Theory so that we have a whole family of truth predicates where before there was just one. These axioms ensure that each state $\mathfrak{w}$ is exhaustive (there is no proposition p such that neither p nor $\hat{\neg} p$ is in $\mathfrak{w}$).

To the language of terms we can add $\mathfrak{W}$, a theory internal analogy of $\mathfrak{w}$:

(PW9) $\mathsf{Pty}(\mathfrak{W})$
(PW10) $\mathsf{True}(\mathfrak{W}(\mathfrak{w})) \leftrightarrow \mathsf{W}(\mathfrak{w})$

Internalised Modal Operators We are now in a position to define modal operators for necessity and possibility. We would like to have the following truth conditions follow from the definition of $\Box$ and $\Diamond$:

$$\mathsf{Prop}(p) \rightarrow (\mathsf{True}(\Box p) \leftrightarrow \forall \mathfrak{w}(\mathsf{W}(\mathfrak{w}) \rightarrow p \in \mathfrak{w}))$$
$$\mathsf{Prop}(p) \rightarrow (\mathsf{True}(\Diamond p) \leftrightarrow \exists \mathfrak{w}(\mathsf{W}(\mathfrak{w}) \wedge p \in \mathfrak{w}))$$

This can be achieved by the following definitions:

(PW11) $\Box p =_{\mathsf{def}} \hat{\forall} \mathfrak{w}(\mathfrak{W}(\mathfrak{w}) \hat{\rightarrow} \mathfrak{w}(p))$
(PW12) $\Diamond p =_{\mathsf{def}} \hat{\exists} \mathfrak{w}(\mathfrak{W}(\mathfrak{w}) \hat{\wedge} \mathfrak{w}(p))$

These definitions naturally allow us to show that $\mathsf{Prop}(\Box p)$ and $\mathsf{Prop}(\Diamond p)$ and that $\mathsf{True}(\Box p) \leftrightarrow \mathsf{True}(\hat{\neg} \Diamond \hat{\neg} p)$ when p is a proposition.

If we had adopted the weaker axioms for $\mathfrak{W}$ above—where $\mathfrak{W}(\mathfrak{w})$ is a proposition only when $\mathfrak{w}$ is of the appropriate category—then we would not be able to prove $\mathsf{Prop}(\Box p)$ and $\mathsf{Prop}(\Diamond p)$. As with $\mathcal{E}(p)$, additional axioms would be required and $\Box, \Diamond$ would no longer be definitional. This is why we adopted the stronger axiomatisation.

The following axiom gives us something akin to necessitation (if $\vdash A$ then $\vdash \Box A$):

(PW13) $\mathsf{Prop}(p) \rightarrow (\mathsf{True}(p) \rightarrow \forall \mathfrak{w}(\mathsf{W}(\mathfrak{w}) \rightarrow p \in \mathfrak{w}))$

Taken together, these axioms give us an **S5** modality.

The theory can be modelled by extending a Frege Structure with a class of terms that correspond with subsets of the class of propositions and supersets of the class of true propositions.

Comments The axioms and definitions above lead to a semantic theory that combines the benefits of fine-grained intensionality, weak typing and first-order power with a treatment of possible worlds. This allows a propositional attitudes to be analysed with the intensionality provided by a Frege Structure, with the appealing possible worlds' treatment of the **S5** metaphysical modalities. In this theory, as in the case of logically equivalent propositions in IHTT, propositions are not equated if they hold in the same sets of possible worlds. In addition, possible worlds are not equated if they contain the same sets of propositions.

If we are just interested in the modalities themselves, there may be more appropriate means of adding them to a theory with fine grained intensionality [31]. There may also be philosophical motivations for having modalities in the language of well formed formulæ (unlike in this account, where they reside exclusively within the language of terms). This would allow us to express the following:

$$\Box\mathsf{Prop}(p) \rightarrow (\mathsf{True}(\Box p) \leftrightarrow \Box\mathsf{True}(p))$$

and the full rule of necessitation:

$$\text{If } \vdash A \text{ then } \vdash \Box A$$

Such a move seems to be required if we are interested in anything other than **S5** modality.

In this paper, we have only presented an explicit formalisation of modalities in the case of Property Theory, and not IHTT. In the latter case, we could proceed by taking IHTT to be our non-modal base logic, and then build on this base logics in which each logic can refer to the prime filters of the previous level. We will explore this approach in future work.

4 Possible Worlds as Prime Filters of Propositions

The propositions (members of the type Π) of the IHTT described in 2 define a bounded distributive prelattice. The axioms (IHTT1)–(IHTT13) yield a Heyting prelattice $L = \langle A, \wedge, \vee, \rightarrow, \top, \bot, \rangle$, while (IHTT1)–(IHTT14) generate a Boolean prelattice $L = \langle A, \wedge, \vee, \neg, \top, \bot, \rangle$, with $A =$ the set of elements of Π. The preorder $\leq$ of each prelattice models an entailment relation that does not satisfy antisymmetry. Therefore, for any proposition p in the prelattice, $\{s : p \leq s \& s \leq p\}$ is an equivalence class whose elements are not necessarily identical.

We can define a pre-lattice for PT corresponding to that given for IHTT in section 2.1, axioms (IHTT1)–(IHTT13), modulo difference in the required typing constraints, where basic propositions are of the form $\mathsf{True}(p)$, and where $a \cong b$ is equivalent to $\mathsf{True}(a) \leftrightarrow \mathsf{True}(b)$ in the case where a, b are propositions, and $\forall x(\mathsf{True}(ax) \leftrightarrow \mathsf{True}(bx))$ in the case where a, b are properties.[9] Alternatively, if we define $\cong$ to be PT's λ-equality (=), and weaken axiom (11) from a

[9] Note that if we take the notion of λ-equality to be basic, then $\top, \bot$ can be defined by $a = a$ and $a \neq a$ respectively.

biconditional to a conditional[10] then we can define a proof-theoretic lattice. A similar move might be possible in IHTT if we take $\cong$ to be the basic equivalence relation.

Following a suggestion from Carl Pollard (presented in [17]) we define an *index* as a prime filter of propositions in the (pre-)lattice generated by IHTT or PT. A prime filter is closed under meet and the (pre-)order relation, contains $p \vee q$ only if it contains p or q, includes $\top$, and excludes $\bot$. Therefore, an index provides a consistent theory closed under entailment. If a prime filter is maximal (i.e. is an ultrafilter), then it partitions the (pre-)lattice into the propositions that it contains and those in the ideal which is its dual. Such an ultrafilter corresponds to a possible world. If the prime filter is non-maximal, then it corresponds to a possible situation which is a proper part of a world. The indices of Heyting (pre-)lattices can be non-maximal situations of this kind, while those of Boolean (pre-)lattices are worlds.

We take a proposition p to be *true at an index* i iff $p \in i$. A logically true (false) proposition is an element of every (no) index defined by the prelattice. Logically equivalent propositions are elements of all and only the same indices of the prelattice. We can introduce meaning postulates on non-logical constants (like axiom (IHTT21) for *most*) to restrict the set of indices to a proper subset of indices that sustain the intended interpretations of these constants. As we have observed in both IHTT and PT, logically equivalent propositions need not be identical. Therefore, they are not intersubstitutable in all contexts, specifically in the complements of verbs of propositional attitude.

By characterizing indices algebraically as prime filters of propositions in (pre-)lattices we are able to dispense with possible worlds and situations as basic elements of our semantic theory. We also characterise truth of a proposition in a world or situation to membership of a prime filter. The fact that we distinguish between identity and equivalence permits us to distinguish between logically equivalent expressions in our semantic representation language.

5 Comparison with Other Hyperintensionalist Approaches

5.1 Intentional Logic

Thomason [28] proposes a higher-order Intentional Logic in which a type p of propositions is added to e (individuals) and t (truth-values) in the set of basic types. The classical truth functions, quantifiers, and identity relation are defined as functions from types τ (in the case of the truth-functions, $\tau = t$ or $\langle t, t \rangle$) into t. A parallel set of intentional connectives, quantifiers, and identity relation are defined as functions from types τ to p. An extensional operator ${}^{\cup}\langle p, t \rangle$ denotes a homomorphism from D_p to 2 (the bounded distributive lattice containing only

[10] That is, if we adopt the axiom $\vdash \forall st(\mathsf{Prop}(s) \wedge \mathsf{Prop}(t) \rightarrow (s = t \rightarrow \mathsf{True}(s) \leftrightarrow \mathsf{True}(t)))$.

1 and 0 ($\top$ and $\bot$). The set of homomorphisms which can provide the interpretation of the extensional operator is constrained by meaning postulates like the following (where $\wedge y$ is the extensional universal quantifier, $\cap y$ is the intentional universal quantifier, $\neg$ is extensional negation, $\sim$ is intentional negation, $\wedge$ is extensional conjunction, $\cap$ is intentional conjunction, $=$ is extensional identity, and $\approx$ is intentional identity).

(1) a. $\wedge y^p({}^\cup\!\sim y = \neg{}^\cup y)$
b. $\wedge y^p z^p({}^\cup(y \cup z) ={}^\cup y \wedge{}^\cup z)$
c. ${}^\cup \cap x^\tau \phi = \wedge x^{\tau\,\cup}\phi$
d. ${}^\cup(\alpha \approx \beta) = (\alpha = \beta)$

(2) $\wedge x^e({}^\cup \mathit{groundhog}'_{\langle e,p\rangle}(x) = \mathit{woodchuck}'_{\langle e,p\rangle}(x))$

This last axiom (2) requires that for any individual a, the truth-value of ${}^\cup\mathit{groundhog}'(a)$ is identical to that of ${}^\cup\mathit{woodchuck}'(a)$. It does not require that $\mathit{groundhog}'(a)$ and $\mathit{woodchuck}'(a)$ be identical propositions, and so it is compatible with (3).

(3) $\wedge x^e \neg(\mathit{groundhog}'(x) = \mathit{woodchuck}'(x))$

The main problem with Thomason's proposal is that he does not specify the algebraic structure of the domain of propositions D_p or the entailment relation which holds among its elements. The connection between intentional identity and intentional bi-implication is not specified, and so the interpretation of these relations is crucially underdetermined.[11]

Consider the theorem (4) (a proof is presented in [14, p.14]), which has (5) as a corollary.

(4) For any a, b in a distributive lattice L such that $a \not\leq b$ there is a homomorphism $h : L \to 2$ in which $h(a) \neq h(b)$.

(5) For any a, b in a distributive lattice L, if every homomorphism $h : L \to 2$ is such that $h(a) = h(b)$, then $a = b$.

If D_p is a bounded distributive lattice, then Thomason must allow the set of homomorphisms from D_p to 2 that provide possible intepretations of the extensional operator ${}^\cup\langle p, t\rangle$ to contain mappings from D_p to 2 which do not respect the meaning postulates that he imposes upon the operator. Such homomorphisms will specify impossible worlds which distinguish between non-identical propositions that the meaning postulates require to be identical in truth-value in the subset of homomorphisms corresponding to the intended interpretations of the elements of D_p.

Alternatively, Thomason could characterize D_p as a prelattice whose preorder is not anti-symmetrical. The fact that he does not provide a proof theory for his domain of propositions that specifies an entailment relation leaves these central

[11] These problems were pointed out by Carl Pollard and discussed in [18].

issues unsettled. Therefore, it is not clear precisely how Thomsason's Intentional Logic permits us to distinguish between logically equivalent expressions.

By contrast, in our framework the proof theory defines a relation between generalized equivalence and identity in which the former does not reduce to the latter. We do not require impossible worlds, as we model the entailment relation of the logic in such a way that logically equivalent propositions remain (possibly) distinct.

5.2 Data Semantics

Landman [16] uses a distributive De Morgan lattice L to to model a first-order language. He identifies *facts* as elementary (non-negated) elements of L. Propositions are constructed by applying the operations of L to facts. According to Landman, if a and b are distinct facts in L, $p = a \vee \neg a$ and $q = b \vee \neg b$, then $p \neq q$ beacuse p and q are generated from different facts.

It is not clear how Landman can sustain this distinction between logically equivalent propositions. If L is a bounded distributive lattice, then all logically true propositions reduce to $\top$, and all logically false propositions are identical to $\bot$ by virtue of the antisymmetry of the partial order relation of the lattice. In general, for any proposition p in a distributive lattice, all propositions that are logically equivalent to p are identical to p.

5.3 Infon Algebras

Barwise and Etchemendy [2] propose an infon algebra as the framework for developing the model theory of situation semantics. An infon algebra $\mathrm{I} = \langle Sit, I, \Rightarrow, \models\rangle$ where *Sit* is a non-empty set of situations, I is a non-empty set of infons, $\langle I, \Rightarrow\rangle$ is a bounded distributive lattice, and $\models$ is a relation on $Sit\ x\ I$ that satisfies the following conditions.

(6) a. If $s \models \sigma$ and $\sigma \Rightarrow \tau$, then $s \models \tau$.
b. $\neg(s \models 0)$ and $s \models 1$.
c. If Σ is a finite set of infons, then $s \models \wedge\Sigma$ iff for each $\sigma \in \Sigma$, $s \models \sigma$.
d. If Σ is a finite set of infons, then $s \models \vee\Sigma$ iff for some $\sigma \in \Sigma$, $s \models \sigma$.

The conditions that 6 imposes on the $\models$ relation require that the set of infons which a situation supports is a prime filter.

$Supports : I \rightarrow Pow(Sit)$ is a homomorphism from the set of infons to the power set of situations such that for each $\sigma \in I$, $Supports(\sigma) = \{s \in Sit : s \models \sigma\}$. Consider the following generalization of Johnstone's theorem 4.

(7) Let L be a bounded distributive lattice. For any a, b in a distributive lattice A, if every homomorphism h: A $\rightarrow$ L is such that $h(a) = h(b)$, then $a = b$.
Proof: It follows from 4 that for any a, b in a distributive lattice A such that $a \not\leq b$ there is a homomorphism g: A $\rightarrow$ 2 in which $g(a) = 1$ and $g(b) = 0$. 2 can be embedded in any bounded distributive lattice L by the

homomorphism f: $2 \to \mathrm{L}$ which is such that $f(1) = 1$ and $f(0) = 0$. Let h: $\mathrm{A} \to \mathrm{L} = f \circ g$. Then for any a, b in a distributive lattice A such that $a \not\leq b$, there is homomorphism $h : A \to \mathrm{L}$ in which $h(a) \neq h(b)$. The theorem follows.

The power set of situations is a bounded distributive lattice. Therefore, if for all homomorphisms $h : I \to Pow(Sit)$ that are permitted instances of *Supports* two infons $\sigma, \tau \in I$ are such that $h(\sigma) = h(\tau)$, then $\sigma = \tau$. But then in order to distinguish between two infons σ and τ which are logically equivalent or equivalent by virtue of meaning postulates it is necessary to posit homomorphisms from I to $Pow(Sit)$ in which $h(\sigma) \neq h(\tau)$. These homomorphisms correspond to impossible situations (impossible partial worlds).

5.4 Situations as Partial Models

Muskens [22] uses a many-valued logic to defined indices as partial models rather than complete worlds. Indices correspond to situations, and they are partially ordered by (distinct types of) containment relations. A partial model M is the indexed union of a set of indices, where each index in the set is an indexed partial model M_i.

Let Γ, Δ be sets of propositions. Δ *strongly follows* from Γ iff in every intended model M (a model that satisfies a given set of axioms), the partial intersection of values of the elements of Γ is included in the partial union of the values of the elements of Δ. For two propositions p, q, q *weakly follows* from p iff for every intended model M, in each M_i of M that satisfies a specified set of meaning postulates, the value of p in M_i is (partially) included in the the value of q at M_i.

Substitution of propositional arguments in predicates denoting relations of propositional attitude is restricted to cases of strong mutual entailment. (8)a and (8)b weakly entail each other.

(8) a. $\mathit{groundhog}'(c)$
b. $\mathit{woodchuck}'(c)$

They are equivalent only in the partial models that satisfy the meaning postulate in 9).

(9) $\lambda i[\forall x(\mathit{groundhog}'(x, i) = \mathit{woodchuck}'(x, i))]$

As (8)a and (8)b are weakly synonymous they are not intersubstitutable in belief contexts.

Muskens' distinction between weak and strong entailment effectively invokes impossible (partial) worlds to distinguish between equivalent propositions.

6 Conclusion

We have presented an axiomatic approach to constructing a hyperintensional semantics for natural language. On this approach we discard the axiom of extensionality and introduce a generalized equivalence (weak identity) relation that does not reduce to strict (logical) identity. We can model the entailment relation of our logic with a bounded distributive (pre-)lattice in which the (pre-)order does not satisfy antisymmetry. We define indices algebraically as prime filters of propositions in the (pre-)lattice, and we reduce truth at an index to membership of such a prime filter. By adding a hyperintensional number theory to our logic we also provide an internal axiomatic treatment of generalized quantifiers like *most* that specify relations among the cardinalities of properties. We have shown how it is possible to implement this approach within both an intensional higher-order type theory and many-sorted first-order Property Theory. Unlike alternative hyperintensionalist models that have been proposed, we can distinguish among equivalent propositions without resorting to impossible worlds to sustain the distinction.

In future work we will extend our axiomatic framework to provide a logic of propositional attitudes, specifically belief. We will also develop a refined characterization of modality that permits us to encode modal logics other than S5. We intend to fully integrate the account of modality given here for PT into our proposed algebraic reduction of possible worlds/situations and to develop a parallel account for IHTT. We hope to implement these hyperintensional treatments of propositional attitudes and modality within a theorem proving module for a computational semantic system.

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